TENTAMEN A RINGEN EN GALOISTHEORIE 21-04-2011

- On each sheet of paper you hand in write your name and student number
- Do not provide just final answers. Prove and motivate your arguments!
- The use of computer, calculator, lecture notes, or books is not allowed
- Use of a single personal A4 sheet of paper is allowed
- In each part of a problem you solve may use previous parts even if you did not prove them

Problem A (20 points)

(1) Prove the correspondence theorem of rings: If I is a proper ideal in a ring R, then there is a bijection from the family of all intermediate ideals J, where $I \subseteq J \subseteq R$, to the family of all ideals in R/I, given by

$$J \mapsto \pi(J) = J/I = \{a + I \mid a \in J\},$$

where $\pi : R \to R/I$ is the natural map. Moreover, if $J \subseteq J'$ are intermediate ideals then $\pi(J) \subseteq \pi(J')$.

(2) Prove that a proper ideal I in a ring R is maximal if, and only if, R/I is a field.

Problem B (20 points)

(1) Let F be a field and $f(x) \in F[x]$ an irreducible polynomial. Let $F \subseteq E$ be a field extension such that E contains an element α for which $f(\alpha) = 0$. Prove that there is a subfield $E' \subseteq E$ and an isomorphism of fields $\psi : F[x]/(f(x)) \to E'$ which satisfies: $\psi(X + (f(x))) = \alpha$.

(2) Prove Kronecker's Theorem: Given a field F and a polynomial $f(x) \in F[X]$ there is a field extension $F \subseteq E$ such that in E the polynomial $f(x)$ splits as a product of linear factors.

Problem C (20 points) Let $f(x) = x^3 - 11 \in \mathbb{Q}[x]$. Let $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{C} - \mathbb{R}$ satisfying $\alpha^3 = 11$ and $\omega^3 = 1$. Let $E \subseteq \mathbb{C}$ be a splitting field of $f(x)$ and let $F = \mathbb{Q}(\alpha)$.

(1) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.
(2) Calculate the minimal polynomial of α over $\mathbb{Q}$.
(3) Prove that $E = \mathbb{Q}(\alpha, \omega)$.
(4) Prove that $[E : \mathbb{Q}] = 6$.
(5) Decompose $f(x)$ in $F[x]$ into irreducible factors.
(6) Calculate the minimal polynomial of $\alpha \cdot \omega$ over F.
(7) Prove that $F(\omega) = F(\omega^2) = E$.
(8) Prove that $Gal(E : \mathbb{Q}) \cong S_3$.
(9) Prove that $Gal(E : F) = \{id, \sigma\}$ where σ satisfies $\sigma(\omega) = \omega^2$.
(10) Prove that there is an element $\tau \in Gal(E : \mathbb{Q})$ for which $\tau(\omega) = \omega^2$.

Problem D (40 points) For each of the following statements decide if it is true or false. Do not provide an argument but do state clearly what your answer is. A correct answer is worth 4 points. An incorrect answer is worth -1 points. A blank answer is worth 0 points.

(1) $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$.
(2) $\mathbb{Q}[X]/(X^2 + 1) \cong \mathbb{C}$.
(3) The subset $A \subseteq \mathbb{C}$ of all complex numbers that are algebraic over $\mathbb{Q}$ is a field.
(4) The subset $A \subseteq \mathbb{R}$ of all real numbers that are algebraic over $\mathbb{Q}$ is a field.
(5) There exists a number $x \in \mathbb{Z}$ such $\mathbb{Z}/(x)$ is a domain but not a field.
(6) $Gal(\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}) = \{id\}$.
(7) The polynomial ring $\mathbb{C}[X, Y, Z]$ is a principal ideal domain. False ?
(8) The polynomial $18 \cdot X^{17} + 4X^{10} - 2X^2 - 1$ is irreducible in $\mathbb{Q}[X]$.
(9) Given fields $E \subseteq F \subseteq G$ it is always the case that $Gal(G : F) \subseteq Gal(G : E)$.
(10) If $f(x) \in \mathbb{Q}[x]$ and $E : \mathbb{Q}$ is a splitting field extension of $f(x)$ then $|Gal(E : \mathbb{Q})| = [E : \mathbb{Q}]$.