Please write clearly and give full proofs of all your claims. This exam is open book, which means you are allowed to use the course material "Number Rings" written by P. Stevenhagen. In your solutions, you may refer to numbered results and examples contained in "Number Rings". Calculators may be used. You are not permitted to use any other tools.

Problem 1. (10 points) Let $\mathcal{O}_K$ be the ring of integers of the quadratic field $K = \mathbb{Q}(\sqrt{-21})$. Compute the unit group $\mathcal{O}_K^\times$ and the class group $\mathrm{Cl}(\mathcal{O}_K)$.

Problem 2. For a number field K, let $\mathcal{O}_K^\times$ denote the unit group of its ring of integers.

- (5 points) Determine all number fields K which have the property
$$\mathcal{O}_K^\times \simeq \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}.$$
- (5 points) Does there exist a number field K which has the property
$$\mathcal{O}_K^\times \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}?$$

Problem 3. (10 points) Find all integer solutions $x, y \in \mathbb{Z}$ to the equation
$$y^2 = x^3 - 11.$$

Problem 4. Let $k > 1$ be an integer, and set $K = \mathbb{Q}(\sqrt{1-4k})$. Assume $1-4k$ is squarefree

(a) Assume that $|\mathrm{Cl}(K)| = 1$.

- (4 points) Prove that if $p < k$ is prime, then p is inert in $K/\mathbb{Q}$.
- (4 points) Prove that the polynomial
$$f := x^2 + x + k \in \mathbb{Z}[x]$$
has the property that $f(n)$ is a prime number, for all $n \in \{0, 1, \ldots, k-2\}$.

(b) (2 points) Show that we have $|\mathrm{Cl}(K)| = 1$ when $k = 41$.