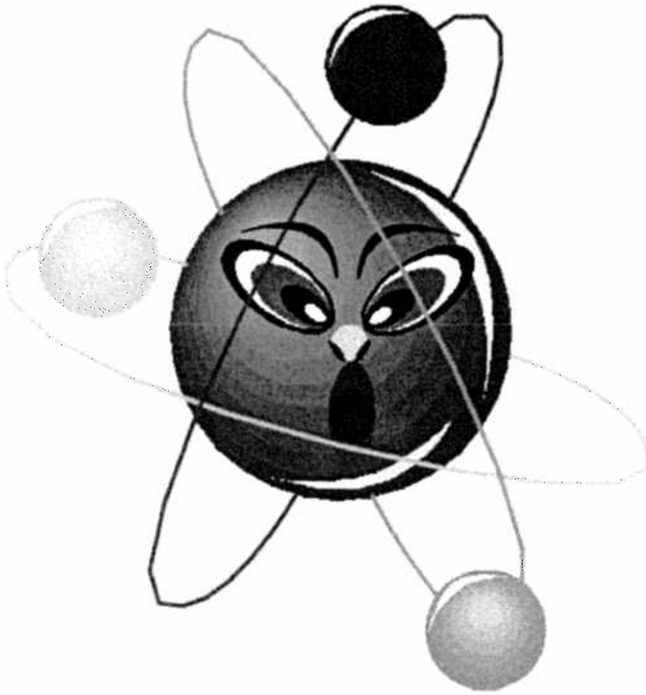
Examination
Experimental Quantum Physics
Part Two



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The Zeeman effect:

The interaction of an atom with the magnetic field is given by

$$\mathcal{H}_Z = -\vec{\mu} \cdot \vec{B},$$

with $\vec{\mu}$ the magnetic dipole moment. For an atom the magnetic moment is given by

$$\vec{\mu} = -\frac{\mu_B(\vec{\ell} + 2\vec{s})}{\hbar},$$

with $\mu_B = e\hbar/2m$ the Bohr magneton, $\vec{\ell}$ the orbital angular momentum and $\vec{s}$ the spin angular momentum. We want to calculate the effect of the Zeeman effect on the transition frequency of the s→p transition in the alkali-metal atoms.

1. Give the quantum numbers ℓ and s of the two states involved.
2. For a strong magnetic field, calculate the Zeeman shift for an electron in the s-state assuming the magnetic field is in the z-direction.
3. For a strong magnetic field, calculate the Zeeman shift for an electron in the p-state assuming the magnetic field is in the z-direction.
4. Find the transition frequencies from the ground state (s-state) to the excited state (p-state) using the selection rules $\Delta m_s = 0$ and $\Delta m_\ell = \pm 1$.
5. The states are split by the spin-orbit interaction, which is given by

$$\mathcal{H}_{\text{SO}} = \frac{\xi(r)\vec{\ell} \cdot \vec{s}}{\hbar^2},$$

   where the function $\xi(r)$ is only a function of the radius r of the electron. What are the possible values of the quantum number j for both the s- and p-state, where $\vec{j}$ is the total angular momentum: $\vec{j} = \vec{\ell} + \vec{s}$. *Hint: In this exercise we will neglect the effect of the nuclear spin I, since the shift caused by the nuclear spin is much smaller compared to the spin-orbit shift.*
6. How large is the spin-orbit shift for the ground state (s-state)?
7. How do the transition frequencies change, if the shift due to the magnetic field is smaller than the shift due to the spin-orbit interaction. Do NOT calculate the frequencies, but provide a qualitative estimate.

Magneto-optical trapping: Consider in 1-D the effect of two counter-propagating laser beams on the atoms. Here the atoms are in a magnetic field $B = \alpha z$, which depends linear on the position z in the trap. The total force on the atoms is given by $F = F_+ + F_-$, where for low intensity the force of the individual beams are given by

$$F_\pm = \pm \frac{\hbar k \gamma}{2} \frac{s_0}{1 + s_0 + (2\delta_\pm/\gamma)^2}$$

and the detuning $\delta_\pm$ for each laser beam is given by

$$\delta_\pm = \delta \mp kv \pm \frac{\mu_B B}{\hbar}.$$

Here μ_B is the magnetic moment for the transition used. Note that the Doppler shift $\omega_D \equiv -kv$ and the Zeeman shift $\omega_Z = \mu_B B/\hbar$ both have opposite signs for opposite beams. Assume that the detuning δ is large compared to the Doppler and Zeeman shift.

1. Calculate that to first order the total force in the center of the trap ($B = 0$) is given by $F = -\beta v$, and derive an expression for the damping coefficient β in terms of the saturation parameter s_0, detuning δ, the linewidth γ and wavevector k.
2. Show for the optimum values $\delta = -\gamma/2$ and $s_0 = 2$ that β is given by $\hbar k^2/2$.
3. Use the analogy between the Doppler and Zeeman shift to find that to first order the force for an atom at rest is given by $F = -\kappa z$, and find the relation between β and the spring constant κ. What should be the sign of the gradient α in order to get a restoring force to the center of the trap. *Hint: Note the difference between the spring constant κ and the wavevector k.*
4. The equation of motion of the atoms is given by

$$m\ddot{z} + \beta\dot{z} + \kappa z = 0,$$

   where the dot indicates the time-derivative. Indicate why the solution of this differential equation is given by a damped harmonic oscillator with damping rate $\Gamma_d = \beta/m$ and oscillation frequency $\omega_{\text{osc}} = \sqrt{\kappa/m}$.
5. Calculate for the optimum conditions under (2) the damping rate Γ_d and the oscillation frequency ω_{osc} using a magnetic field gradient of $|\alpha|$=10 G/cm and show using your result that the motion is strongly overdamped.

6. For an overdamped harmonic oscillator the restoring time to the origin τ is given by $\tau = 2\Gamma_d/{\omega_{\rm osc}}^2$. Calculate τ and discuss why the restoring time τ is much longer than the damping time Γ_d.

Important numbers: $m_{\rm Na} = 23$ amu, 1 amu $= 1.67 \times 10^{-27}$ kg, $\mu_B = 9.27 \times 10^{-28}$ J/G and $\hbar = 1.05 \times 10^{-34}$Js.

Bose-Einstein condensation: The distribution function for bosons is given by

$$f_{\mathrm{BE}}(\epsilon) = \frac{1}{\exp[(\epsilon-\mu)/k_B T] - 1}.$$

1. Discuss the chemical potential μ. What are the limits of μ for an atomic, non-interacting Bose gas? Discuss the physical reasons for these limits.
2. For an atomic Bose gas the density of states is given by

$$\rho(\epsilon) = \rho(k)\frac{\mathrm{d}k}{\mathrm{d}\epsilon} = \frac{V}{4\pi^2\hbar^3}\sqrt{8m^3\epsilon}.$$

   Calculate the number of atoms at a certain temperature T in a volume V. *Hint: Use the definite integral given below this exercise to evaluate your integral.*
3. Show that this number is limited and what happens to the excess number of atoms.
4. Use your result to derive an expression for the critical temperature T_c, where the number of atoms is equal to the maximum number of atoms in the gas.
5. Calculate the total energy of the gas, assuming the temperature of the gas is below T_c.
6. Calculate the heat capacity $C_V = \partial E/\partial N$ at constant volume V below T_c.

The Bose-Einstein functions $g_\alpha(z)$ with $0 \leq z \leq 1$ is defined as

$$g_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}.$$

This allows to write

$$\int_0^\infty \mathrm{d}x\, \frac{x^{\gamma-1}}{z^{-1}e^x - 1} = \Gamma(\gamma) g_\gamma(z),$$

where Γ is the usual gamma-function. For $z = 1$ this reduces to

$$\int_0^\infty \mathrm{d}x\, \frac{x^{\gamma-1}}{e^x - 1} = \Gamma(\gamma)\zeta(\gamma).$$

For various values of γ the values are

γ	$\Gamma(\gamma)$	$\zeta(\gamma)$
1	1	∞
$3/2$	$\sqrt{\pi}/2$	2.612
2	1	$\pi^2/6$
$5/2$	$3\sqrt{\pi}/4$	1.341
3	2	1.202
$7/2$	$15\sqrt{\pi}/8$	1.127
4	6	$\pi^4/90$