Institute of Mathematics, Faculty of Mathematics and Computer Science, UU.
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In 2003/2004, the course WISB342 was given by Eduard Looijenga.

Differentieerbare variëteiten (WISB342)
1 november 2003

You may do this exam either in Dutch or in English. Books or notes may not be consulted. Be sure to put your name on every sheet you hand in.
All maps and manifolds are assumed to be C^∞ unless the contrary is explicitly stated.

Question 1

Let $f : M \to N$ and $g : N \to P$ be maps between manifolds.

a) Prove that if f and g are submersions, then so is gf.

b) Prove that if f and g are immersions, then so is gf.

c) Prove that if f and g are embeddings, then so is gf. (Hint: you may use the fact that an embedding is an immersion which also a homeomorphism onto its image.)

Question 2

a) Prove that the tangent bundle of S^3 is trivial.

b) Is the tangent bundle of P^3 trivial?

Question 3

The Möbius strip M can be defined as follows: let $M_0 := (-\pi, \pi) \times (-1, 1)$ and $M_1 := (0, 2\pi) \times (-1, 1)$ and identify the open subset $U_0 := ((-\pi, \pi) - \{0\}) \times (-1, 1)$ of M_0 with the open subset $U_1 := ((0, 2\pi) - \{\pi\}) \times (-1, 1)$ of M_1 by means of the diffeomorphism $h : U_0 \to U_1$

$$h(x, y) = \begin{cases} (x + 2\pi, -y) \in U_1 & \text{in case } x \in (-\pi, 0); \\ (x, y) \in U_1 & \text{in case } x \in (0, \pi). \end{cases}$$

You may assume that M is a is a Hausdorff space and that the inverses of the maps $M_0 \to M$, $M_1 \to M$ define an atlas.

a) Prove that there is a vector field V on M whose restriction to M_0 resp. M_1 is given by $\partial/\partial x$.

b) Prove that V generates a flow $H : \mathbb{R} \times M \to M$ on M. Describe this flow in terms of the coordinates (x, y) on M_0 and M_1. Show that $H_{4\pi}$ is the identity map, but that $H_{2\pi}$ is not.

c) Explain why M is not orientable.

d) Let $N \subset M$ be the complement of the central circle (so where $y \neq 0$ on both M_0 and M_1). Prove that N is diffeomorphic to the open cylinder $S^1 \times (0, 1)$. Is N orientable?

Question 4

Let $H : \mathbb{R} \times M \to M$ be a flow on an m-manifold M and let the vector field V be its infinitesimal generator.

a) Let $f : M \to \mathbb{R}$. Prove that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} H_t^* f = V(f).$$

b) Let W be a vector field on M. Prove that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} H_t^* W = [V, W].$$

c) Suppose that M is oriented. Prove that for every m-form μ on M with compact support, the integral $\int_M H_t^* \mu$ is independent of t.