Differentiable manifolds 2016-2017: Final Exam

Notes:

1. **Write your name and student number** **clearly** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.
7. Every individual question is worth 10 points, giving a total of 140 points for the entire exam.

Questions

Exercise 1(30 pt) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}$ given by $F(x, y, z) := x^2 + y^2 - z^2$.

a) For which $c \in \mathbb{R}$ is $M_c := F^{-1}(c)$ a smooth submanifold of $\mathbb{R}^3$? Give a sketch of M_c for all $c \in \mathbb{R}$.

b) Show that M_1 is diffeomorphic to $S^1 \times \mathbb{R}$ and that M_{-1} is diffeomorphic to $\mathbb{R}^2 \coprod \mathbb{R}^2$.

c) Construct an atlas for M_1 and compute the transition maps.

Exercise 2(20 pt)

a) Let V and W be vector spaces and $L : V \to W$ a linear map. Recall that the rank of L is the dimension of its image $L(V) \subset W$. Show that the rank of L is the biggest number k for which $\Lambda^k L : \Lambda^k V \to \Lambda^k W$ is nonzero.

b) For a nonzero vector $v \in V$ we consider for each $k \geq 0$ the linear map $v\wedge : \Lambda^k V \to \Lambda^{k+1} V$ given by $\alpha \mapsto v \wedge \alpha$. Show that its kernel is given by the image of $v\wedge : \Lambda^{k-1} V \to \Lambda^k V$. (Hint: construct a convenient basis for V.)

Exercise 3(30 pt) Consider the two-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $\mathbb{R}^3$.

a) Compute $\int_{S^2(r)} \omega$, where $S^2(r) := \{(x, y, z) | x^2 + y^2 + z^2 = r^2\}$ is the two-sphere of radius $r > 0$ in $\mathbb{R}^3$.

b) Let $\alpha := f \cdot \omega \in \Omega^2(\mathbb{R}^3 \backslash 0)$ where f is the function given by $f(x, y, z) := (x^2 + y^2 + z^2)^{-\frac{3}{2}}$. Show that $d\alpha = 0$ and use this to conclude that $\int_{S^2(r)} \alpha$ is independent of $r \in \mathbb{R}_{>0}$. What is its value?

c) Let V be the vector field on $\mathbb{R}^3 \backslash 0$ given by $V_{(x,y,z)} := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Compute the flow φ_t^V of V and show that $(\varphi_t^V)^* \alpha = \alpha$. Use this to give another proof of the fact that $\int_{S^2(r)} \alpha$ is independent of r.

Exercise 4(30 pt) For this exercise you may use without proof that $\int_{S^n} : H^n(S^n) \to \mathbb{R}$ is an isomorphism. Let $\pi : S^n \to \mathbb{RP}^n$ denote the quotient map and $\iota : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the antipodal map $x \mapsto -x$.

a) Show that a form $\omega \in \Omega^k(S^n)$ is of the form $\omega = \pi^*\alpha$ for a unique $\alpha \in \Omega^k(\mathbb{RP}^n)$ if and only if $\iota^*\omega = \omega$. Deduce that $\frac{1}{2}(\omega + \iota^*\omega) \in \pi^*(\Omega^k(\mathbb{RP}^n))$ for every $\omega \in \Omega^k(S^n)$.

b) If n is even and $\iota^*\omega = \omega$, show that $\int_{S^n} \omega = 0$.

c) Show that $H^n(\mathbb{RP}^n) = 0$ for all even n. Deduce that $\mathbb{RP}^n$ is not orientable for n even. (Hint: for $\omega \in \Omega^n(\mathbb{RP}^n)$ show that $\pi^*\omega$ is exact. Then use part a) to write $\pi^*\omega = d\alpha$ for some α with $\iota^*\alpha = \alpha$.)

Exercise 5(30 pt) Recall that a vector bundle $\pi : E \to M$ is called orientable if we can choose an orientation on each fiber, in such a way that around each point in M we can find a positively oriented frame.

a) Show that a line bundle (i.e. a vector bundle of rank 1) is trivial if and only if it is orientable.

b) Show that for any line bundle E over M the line bundle $E \otimes E$ is trivial.

c) Show that the Möbius bundle over S^1 is not trivial.