Geometry and Topology – Exam 3

Notes:

1. **Write your name and student number *****clearly***** on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Questions

Exercise 1 (2.0 pt)**.** For each of the following pairs of spaces, decide if they are homeomorphic or not

a) $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$;

b) $\mathbb{C}P^1$ and S^2;

c) $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$;

d) $\mathbb{C}P^n$ and $\mathbb{R}P^{2n}$;

e) $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$ and $\mathbb{R}P^2 \# T^2$.

Remember to justify your answers.

Exercise 2 (2.0 pt)**.**

a) Let X and Y be path connected. Show that the join $X * Y$ is simply connected;

b) Using a) or otherwise show that if X is path connected and each connected component of Y is path connected then the joint $X * Y$ is simply connected.

Exercise 3 (2.0 pt)**.** Given maps $X \xrightarrow{\pi_1} Y \xrightarrow{\pi_2} Z$ such that both $\pi_2 : Y \longrightarrow Z$ and the composition $\pi_2 \circ \pi_1 : X \longrightarrow Z$ are covering spaces, show that if Z is locally path connected then $\pi_1 : X \longrightarrow Y$ is a covering space.

Exercise 4 (2.0 pt)**.** Let $x_0 \in S^1$ be a fixed point and let X be the quotient space of $S^1 \times S^1$ under the identification $(x_0, y) \sim (x_0, z)$ for all $y, z \in S^1$. Compute all the homology groups of X.

Exercise 5 (2.0 pt)**.** The mapping torus of a homeomorphism $f : X \longrightarrow X$, denoted by Tf, is the quotient of $X \times I$ obtained by identifying $(x, 0)$ with $(f(x), 1)$ for all $x \in X$. Show that we have a short exact sequence

$$\{0\} \longrightarrow \frac{H_i(X)}{\mathrm{Im}(\mathrm{Id} - f_* : H_i(X) \longrightarrow H_i(X))} \longrightarrow H_i(Tf) \longrightarrow \ker(\mathrm{Id} - f_* : H_{i-1}(X) \longrightarrow H_{i-1}(X)) \longrightarrow \{0\}.$$