Measure and Integration: Final 2014-15

(1) Consider a measure space $(X, \mathcal{A}, \mu)$, and let $(f_n)_n$ be a sequence in $\mathcal{L}^2(\mu)$ which is bounded in the $\mathcal{L}^2$ norm, i.e. there exists a constant $C > 0$ such that $||f_n||_2 < C$ for all $n \geq 1$.

(a) Prove that $\sum_{n=1}^{\infty}(\frac{f_n}{n})^2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$. (1 pt.)

(b) Prove that $\lim\limits_{n\to\infty} \dfrac{f_n}{n} = 0$ μ a.e. (1 pt.)

(2) Let $(X, \mathcal{A}, \mu)$ be a finite measure space. Suppose that the real valued functions $f_n, g_n, f, g \in \mathcal{M}(\mathcal{A})$ $(n \geq 1)$ satisfy the following:

(i) $f_n \xrightarrow{\mu} f$,

(ii) $g_n \xrightarrow{\mu} g$,

(iii) $|f_n| \leq C$ for all n, where $C > 0$.

Prove that $f_n g_n \xrightarrow{\mu} fg$. (2 pts)

(3) Let $(X, \mathcal{A})$ be a measurable space and let μ, ν be finite measures on $\mathcal{A}$.

(a) Show that there exists a function $f \in \mathcal{L}^1_+(\mu) \cap \mathcal{L}^1_+(\nu)$ such that for every $A \in \mathcal{A}$, we have

$$\int_A (1 - f)\, d\mu = \int_A f\, d\nu.$$

(1 pt)

(b) Show that the function f of part (a) satisfies $0 \leq f \leq 1$ μ a.e. (1 pt)

(4) Let $0 < a < b$. Prove with the help of Tonelli's theorem (applied to the function $f(x, t) = e^{-xt}$) that $\int_{[0,\infty)}(e^{-at} - e^{-bt})\frac{1}{t}d\lambda(t) = \log(b/a)$, where λ denotes Lebesgue measure. (2 pts)

(5) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $f \in \mathcal{M}(\mathcal{A})$ satisfies $f^n \in \mathcal{L}^1(\mu)$ for all $n \geq 1$.

(a) Show that if $\lim_{n\to\infty} \int f^n\, d\mu$ exists and is finite, then $|f(x)| \leq 1$ μ a.e. (1 pt)

(b) Show that $\int f^n\, d\mu = c$ is a constant for all $n \geq 1$ if and only if $f = \mathbf{1}_A$ μ a.e. for some measurable set $A \in \mathcal{A}$. (1 pt)