Mathematisch Instituut, Faculteit Wiskunde en Informatica, UU.
In electronische vorm beschikbaar gemaakt door de TBC van A–Es^2.
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Complexe Functies (COMPL)
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Write your name and student number on every page you hand in.
Prove your assertions.

Opgave 1

Let f be a holomorphic function on a connected open subset $\Omega \subset \mathbb{C}$. Denote by $\Omega' \subset \mathbb{C}$ the image of Ω under complex conjugation.

a. Suppose that the function $z \in \Omega \mapsto \overline{f(z)}$ is also holomorphic. Prove that f is constant.

b. Suppose that the function $z \in \Omega' \mapsto f(\bar{z})$ is also holomorphic. Prove that f is constant.

c. When is the function $z \in \Omega' \mapsto \overline{f(\bar{z})}$ holomorphic?

Opgave 2

a. Let a, b, c, d be real numbers with $ad - bc > 0$. Prove that the fractional linear transformation $z \mapsto (az+b)(cz+d)^{-1}$ maps the upper half plane (defined by $\mathrm{Im}(z) > 0$) onto itself.

b. Prove also that any fractional linear transformation with that property is of this form.

c. Show that even every holomorphic diffeomorphism f of the upper half plane onto itself is of this form.
(Hint: if $f(i) = a + bi$ with a, b real and $b > 0$, then $z \mapsto b^{-1}(f(z) - a)$ is a holomorphic diffeomorphism of the upper half plane onto itself which fixes i. Now use the Cayley transformation $\tau(z) = (z-i)/(z+i)$ to convert the problem into one which we can solve on the unit disc.)

Opgave 3

Determine the annulus of convergence of the Laurent series

$$\sum_{n \in \mathbb{Z} - \{0\}} \left(\frac{z}{\sqrt{|n|}} \right)^n .$$

Opgave 4

Let $f(z) := z^{-2}(\sin z)^{-1}$.

a. Determine the poles of f and compute the residue at each of these.

b. Evaluate the series $\sum_{n=1}^{\infty} (-1)^n n^{-2}$.

Opgave 5

Show that for $0 < \alpha < 1$ the integral

$$\int_0^\infty \frac{z^\alpha}{1+z^2} dz$$

converges and evaluate its value.