Midterm for Cosmology (ns-tp430m)

Apr 17 at 9am-12pm 2014 in GAMMA of Educatorium (30 points, 30%)

Please solve each problem on a separate sheet of paper and write your name and student number on each sheet! You may use a calculator.

1. Theoretical questions. (*6 points*) Provide concise answers to the following questions:

(a) (*1.5 points*) Name two gravitational tests that test general relativity that involve rotating bodies. (These effects are due to rotation, and they are not present in Newton's theory.)

(b) (*1.5 points*) Name two pieces of evidence for Universe's homogeneity on large scales.

(c) (*1.5 points*) Bicep 2 has recently observed a parity violating (B-mode) polarization in the CMB photons of a primordial origin. According to the standard cosmological theory, what causes these parity violating polarization effects?

(d) (*1.5 points*) Define the flatness problem (for our Universe)!

2. Gravitational redshift. (*4 points*)

An asymptotic observer is observing photons that come from an accretion of a Schwarzschild (non-rotating) black hole, whose metric is $g_{\mu\nu} = \mathrm{diag}((1+2\phi/c^2), -(1+2\phi/c^2)^{-1}, -r^2, -r^2\sin^2(\theta))$, with $\phi = -G_N M/r$, and is wondering how close to the event horizon photons must originate in order to exhibit a redshift z. Show that the answer to this question is,

$$\frac{\delta r}{r_S} \equiv \frac{r - r_S}{r_S} = \frac{1}{z(z+2)}\,, \tag{1}$$

where $r_S = (2G_N M)/c^2$ is the Schwarzschild radius of the black hole. Calculate δr for a black hole with radius $r_S = 10$ km and the redshift corresponding to photon decoupling, $z = z_{\rm dec} = 1090$. Estimate the redshift exhibited by photons in the Pound-Rebka experiment. This problem illustrates how strong is the gravitational field in an expanding universe.

Hint: You may use that the photons in the Pound-Rebka experiment climbed about 100 m in a gravitational field $\phi_N \simeq 10^{-9}c^2$. The Earth's radius is about $R_E \simeq 6370$ km.

3. Kination in a spatially curved universe. (*7 points*)

Consider a massless homogeneous scalar field $\phi(t)$, whose action is given by,

$$S_\phi = \int d^4x\sqrt{-g}\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu}\,, \tag{2}$$

where $g = \det[g_{\mu\nu}]$ and $g_{\mu\nu} = \mathrm{diag}\Big(1, -a^2/(1-\kappa r^2), -a^2r^2, -a^2r^2\sin^2(\theta)\Big)$.

(a) (2 points) Show that the energy density, pressure and equation of motion for the scalar field are given by,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2\,; \quad \mathcal{P}_\phi = \frac{1}{2}\dot{\phi}^2\,; \quad \ddot{\phi} + 3H\dot{\phi} = 0\,. \tag{3}$$

(b) (3 points)

Solve the corresponding Friedmann equation for the scale factor in conformal time, $a = a(\eta)$. Consider both cases $\kappa > 0$ and $\kappa < 0$, *i.e.* show that

$$a(\eta) = a_0\Big[\sin(2c\sqrt{\kappa}\eta)\Big]^{\frac{1}{2}}\ (\kappa{>}0)\,, \quad a(\eta) = a_0\Big[\sinh(2c\sqrt{-\kappa}\eta)\Big]^{\frac{1}{2}}\ (\kappa{<}0)\,, \quad a_0^4 = \frac{8\pi G_N\rho_0}{3c^4|\kappa|}\,. \tag{4}$$

Hint: The following integrals you may find useful: $\int\frac{dx}{\sqrt{1+x^2}} = \mathrm{Arsinh}(x)$, $\int\frac{dx}{\sqrt{1-x^2}} = \mathrm{Arcsin}(x)$.

(c) (2 points) Sketch the corresponding conformal diagrams for $\kappa > 0$ and for $\kappa < 0$. Discuss in particular in which case(s) Big Bang and Big Crunch singularities occur.

4. The age of the Universe. *(6 points)*

(a) *(2 points)* The Hubble parameter today is $H_0 \simeq 68$km/s/Mpc. Express H_0 in inverse seconds and in giga-electron volts (GeV), *i.e.* show that $H_0 \simeq 2.20\times10^{-18}\ \mathrm{s}^{-1} \simeq 3.35\times10^{-35}\ \mathrm{eV}\hbar^{-1}$. Estimate the age of the Universe as $t_0 \simeq 1/H_0$ (express it in giga-years, Gy).

Hint: Use the following unit conversions: $1\ \mathrm{Mpc} = 3.0857\times10^{19}$ km, $1\ \mathrm{s} = 1.51927\times10^{15}\ \hbar/\mathrm{eV}$.

(b) *(4 points)*

In order to find out a more precise value of the age of the Universe, solve the Friedmann equation,

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G_N}{3c^2}\rho_{\mathrm{m}} + \frac{\Lambda}{3} \tag{5}$$

and show by integrating equation (5) that the scale factor reads

$$a^3(t) = \frac{8\pi G_N}{c^2\Lambda}\rho_{\mathrm{m}0}\sinh^2\left(\frac{\sqrt{3\Lambda}\,t}{2}\right), \tag{6}$$

where $a(0) = 0$, $a(t_0) = 1$, $\rho_{\mathrm{m}}(t) = \rho_{\mathrm{m}0}/a(t)^3$ (nonrelativistic matter). By inverting this expression, show that the age of the Universe can be written as,

$$t_0 = \frac{1}{3H_0}\frac{1}{\sqrt{\Omega_\Lambda}}\ln\left[\frac{1+\sqrt{\Omega_\Lambda}}{1-\sqrt{\Omega_\Lambda}}\right], \tag{7}$$

where $\Omega_\Lambda = \Lambda/(3H_0^2)$. Do you get the expected result in the limit when $\Omega_\Lambda \to 0$? What is t_0 for the Planck value $\Omega_\Lambda = 0.69 \pm 0.02$? How does your result compare with the age of the Universe quoted by the Planck collaboration, $t_0 = 13.8 \pm 0.1$ Gy?

Hint: The following integrals you may find useful: $\int \frac{dx}{\sqrt{1+x^2}} = \text{Arsinh}(x)$.

5. Particle horizons. *(7 points)*

The line element of space times with a constant curvature can be written as,

$$ds^2 = c^2 dt^2 - a^2 \left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega_2^2 \right] , \tag{8}$$

where $d\Omega_2^2 = d\theta^2 + \sin^2(\theta) d\phi^2$ is the surface element of the two-sphere S^2 ($\theta \in [0,\pi], \phi \in [0, 2\pi)$), $a = a(t)$ denotes the scale factor, and $\kappa > 0, \kappa = 0, \kappa < 0$ for a positively curved, flat or negatively curved geometry, respectively.

Assume a universe that begins in an inflationary epoch with $\epsilon = (d/dt)(1/H) =$ const. and $0 < \epsilon \ll 1$, which is followed by a radiation era (that begins at $H_{\text{i-r}} \simeq 10^{13}$ GeV$/\hbar$), continues at a redshift $z_{\text{eq}} \simeq 3300$ as a matter era up to today (for simplicity neglect the late dark energy dominated era). You can make use of a sudden matching approximation, that is you can assume that after a sudden end, inflation is followed by a radiation era, which suddenly converts into a matter era.

(a) *(2 points)* Show first that the redshift at the end of inflation is, $z_{\text{i-r}} \simeq 1.31 \times 10^{29}$.

Hint: Assume that at all times the universe's evolution can be well approximated by one perfect fluid, and that it is in thermal equilibrium at all times during radiation era, which means that $H_{i-r} = [(1+z_{\text{i-r}})^2/(1+z_{\text{eq}})^2] H_{\text{eq}}$. Also, use unit conversions from problem **4(a)** of this exam.

(b) *(3 points)* By assuming for simplicity $\kappa = 0$, show next that in the corresponding eras the curvature scales as,

$$\left(\frac{\Omega_\kappa(t)}{\Omega_\kappa(t')}\right)_{\text{infl}} = \left(\frac{a(t')}{a(t)}\right)^{2(1-\epsilon)} ; \quad \left(\frac{\Omega_\kappa(t)}{\Omega_\kappa(t')}\right)_{\text{rad}} = \left(\frac{a(t)}{a(t')}\right)^2 ; \quad \left(\frac{\Omega_\kappa(t)}{\Omega_\kappa(t')}\right)_{\text{matt}} = \frac{a(t)}{a(t')} . \tag{9}$$

Sketch these curves on a $\ln(\Omega_\kappa) - \ln(a)$ diagram. Make sure to make them continuous at the transition between different eras.

(c) *(2 points)* Based on the above results, calculate by how many time the size of the universe needs to increase during inflation in order to solve the flatness problem. Assume that today $\Omega_\kappa(t_0) = 0.01$.

Hint: The flatness problem is considered solved if $\Omega_\kappa = 1$ at the beginning of inflation.

Cosmology Formulae Midterm 2013-14

Metric compatibility, covariant derivative and the (Levi-Cività) connection $\Gamma^{\alpha}_{\mu\nu}$:

$$\nabla_\alpha g_{\mu\nu} = 0\,,\ \nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^{\mu}_{\alpha\beta}\omega_\mu\,,\ \nabla_\alpha A^\mu = \partial_\alpha A^\mu + \Gamma^{\mu}_{\alpha\beta}A^\beta\,,\ \Gamma^{\alpha}_{\mu\nu} = g^{\alpha\beta}\Big(\partial_{(\mu}g_{\nu)\beta} - \frac{1}{2}\partial_\beta g_{\mu\nu}\Big) \quad (1)$$

Geodesic equation and geodesic deviation (λ is an affine parameter):

$$\frac{Du^\mu}{d\lambda} + \Gamma^{\mu}_{\alpha\beta}u^\alpha u^\beta = 0\,,\ u^\alpha = \frac{dx^\alpha}{d\lambda}\,, \qquad \frac{D}{d\lambda}\frac{d\xi^\mu}{d\lambda} = \mathcal{R}^{\mu}_{\alpha\beta\gamma}u^\alpha u^\beta \xi^\gamma \quad (2)$$

Einstein's equations:

$$G_{\mu\nu} - \frac{\Lambda}{c^2}g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}\,, \qquad G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} \quad (3)$$

Riemann and Ricci tensors (for a symmetric – or Levi-Cività – connection):

$$\mathcal{R}^{\alpha}_{\ \beta\gamma\delta} = 2\partial_{[\gamma}\Gamma^{\alpha}_{\delta]\beta} + 2\Gamma^{\alpha}_{\mu[\gamma}\Gamma^{\mu}_{\delta]\beta}\,, \quad \mathcal{R}_{\alpha\beta} = \mathcal{R}^{\gamma}_{\ \alpha\gamma\beta}\,, \quad \mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta} \quad (4)$$

Covariant actions (Hilbert-Einstein and matter):

$$S_{\rm HE} = -\frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}\left(\mathcal{R} + 2\frac{\Lambda}{c^2}\right)\,, \quad S_{\rm matter} = \int d^4x\sqrt{-g}\mathcal{L}_{\rm matter}(\psi_{\rm matter}, g_{\alpha\beta}) \quad (5)$$

Einstein's equation and matter field equations are obtained by the variation principles:

$$\frac{\delta(S_{\rm HE} + S_{\rm matter})}{\delta g^{\mu\nu}} = 0\,, \quad \frac{\delta S_{\rm matter}}{\delta \psi_{\rm matter}} = 0. \quad (6)$$

Matter actions (scalar, vector, fermionic) $[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \nabla_\mu\psi = (\partial_\mu - \Gamma_\mu)\psi]$:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} - V(\phi) - \frac{\xi}{2}\mathcal{R}\phi^2\,, \quad \mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\,, \quad \mathcal{L}_\psi = \bar\psi \imath\gamma^\mu\nabla_\mu\psi - m_\psi\bar\psi\psi \quad (7)$$

NB: Spin(or) connection Γ_μ is determined by the compatibility condition ($\gamma^\mu = e^{\mu}_{\ a}\gamma^a$):

$$\nabla_\mu\gamma_\nu = \partial_\mu\gamma_\nu - \Gamma^{\alpha}_{\mu\nu}\gamma_\alpha - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0 \quad (8)$$

Stress-energy tensor (general and perfect fluid):

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}\,, \quad (T_{\mu\nu})_{\rm perf.\,fluid} = (\rho + \mathcal{P})\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\mathcal{P} \quad (9)$$

Gravitational dilatation and redshift (cosmological redshift: $z(t) = (a_0/a(t)) - 1$): redshift

$$\frac{\delta t_1}{\delta t_2} \overset{?}{=} \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} \simeq 1 + \frac{\phi_N(r_2) - \phi_N(r_1)}{c^2}\,, \quad \frac{E_1}{E_2} = \frac{\nu_1}{\nu_2} = \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} \simeq 1 + \frac{\phi_N(r_2) - \phi_N(r_1)}{c^2} \quad (10)$$

? **Light deflection:**

time dilation

$$\vec\alpha = -\frac{2}{c^2}\int_{\rm source}^{\rm observer} d\lambda \nabla_\perp \phi_N \longrightarrow \frac{-2}{c^2}\int_{-\infty}^{\infty} dz\ \partial_\perp \phi(d,0,z) \quad (11)$$

FLRW metric (conformal time: $d\eta = dt/a$, Gauss' curvature $R_c = 1/\sqrt{|\kappa|}$):

$$ds^2 = c^2dt^2 - a^2(t)\Big(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\Big) \tag{12}$$

FLRW (Friedmann) equations ($H = (d/dt)\ln(a)$, $a(t_0) = a_0 = 1$):

$$H^2 = \frac{8\pi G_N}{3c^2}\rho + \frac{\Lambda}{3} - \frac{c^2\kappa}{a^2}\,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\rho + 3\mathcal{P}) + \frac{\Lambda}{3} \tag{13}$$

These come together with a conservation equation, and another form of the 2nd equation:

$$\dot{\rho} + 3H(\rho + \mathcal{P}) = 0\,, \quad \dot{H} = -\frac{4\pi G_N}{c^2}(\rho + \mathcal{P}) + \frac{c^2\kappa}{a^2} \tag{14}$$

The principal slow roll parameter ϵ and the EoS parameter w:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right)\,, \quad \text{when } \Lambda = 0 = \kappa:\ \epsilon = \frac{3}{2}(1+w)\,, \quad w = \frac{\mathcal{P}}{\rho} \tag{15}$$

(Physical) particle horizon ($ds = 0$) [comoving horizon $\ell_c = \ell_{\rm phys}/a$]:

$$\ell_{\rm phys} = \int_{r_{\rm in}}^{r}\sqrt{g_{rr}(r')}dr' = a(t)\int_{r_{\rm in}}^{r}\frac{dr'}{\sqrt{1-\kappa r'^2}} = ac(\eta - \eta_{\rm in}) \tag{16}$$

NB: $r_{\rm in}$ = initial radius (may be zero); Hubble radius: $R_H = c/H$, Hubble time: $t_H = 1/H$.

Friedmann equation and relative densities Ω_i for today ($t = t_0$, $H_0 = H(t_0)$):

$$1 = \sum_i \Omega_i + \Omega_\Lambda + \Omega_\kappa\,, \quad \Omega_i = \frac{\rho_i}{\rho_{\rm cr}}\,, \quad \rho_{\rm cr} = \frac{3c^2}{8\pi G_N}H_0^2\,, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}\,, \quad \Omega_\kappa = -\frac{c^2\kappa}{H_0^2}\,\frac{1}{a^2} \tag{17}$$

The age of the Universe (in conformal time: multiply the integrand by $1/\tilde{a}$):

$$tH_0 = \int_0^a \frac{d\tilde{a}}{\sqrt{\Omega_{\rm m}\tilde{a}^{-1} + \Omega_\gamma\tilde{a}^{-2} + \Omega_\Lambda\tilde{a}^2 + \Omega_\kappa + \Omega_Q\tilde{a}^{-1-3w_Q}}}\,, \tag{18}$$

Apparent and absolute magnitudes; luminosity distance d_L:

$$m - M = 5\log_{10}\left(\frac{r}{\rm Mpc}\right) + 25\,, \quad \mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2} \tag{19}$$

Luminosity distance in various geometries:

$$d_L(z) = (1+z)R_c\sinh\Big(\frac{c}{H_0R_c}\int_0^z\frac{dz'}{E(z')}\Big) \quad \text{(open universe)} \tag{20}$$

$$d_L(z) = (1+z)\frac{c}{H_0}\int_0^z\frac{dz'}{E(z')} \quad \text{(flat universe)} \tag{21}$$

$$d_L(z) = (1+z)R_c\sin\Big(\frac{c}{H_0R_c}\int_0^z\frac{dz'}{E(z')}\Big) \quad \text{(closed universe)} \tag{22}$$

with (here Ω_i are defined today at $t = t_0$)

$$E(z)^2 = \Omega_{\rm m}(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_\Lambda + \Omega_Q(1+z)^{3(1+w_Q)} + \Omega_\kappa(1+z)^2\,. \tag{23}$$