Faculteit Natuur- en Sterrenkunde
BOZ/Julius Instituut

Tentamenvoorblad

(gaarne zo volledig mogelijk invullen)

vak: NS-TP401M Quantum Field Theory

tentamennr.*: 2009/20010 – 64

d.d.: 5 november 2009

van 9.00 uur tot 11.00 uur

in gebouw*: Minnaertgebouw zaal*: kantine

Maak de opgave op een apart vel papier!

bijzonderheden:

The use of auxiliary materials such as books, notes, calculators, laptops etc. is NOT permitted.

Please hand in all sheets you used for calculations

*) wordt door BOZ ingevuld

Quantum Field Theory 2009/10 – Midterm Exam, 5 Nov 2009, 9:00-11:00h

You must obtain at least 10 of the 20 points to pass the exam. The use of auxiliary materials such as books, notes, calculators, laptops etc. is NOT permitted. Please hand in all sheets you used for calculations.

Problem 1 - Working with a vector field theory (11 points)

Consider the one-parameter family of Lagrangians

$$\mathcal{L}_\zeta = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\zeta}{2}(\partial_\mu A^\mu)^2, \tag{1}$$

where, as usual, $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$.

(a) Derive the corresponding classical equations of motion for the gauge potential A_μ, and show that they all give rise to the same field equations if the additional gauge condition $\partial_\mu A^\mu = 0$ is imposed at the level of the equations of motion. (*1 point*)

(b) Show that the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A^\nu)(\partial^\mu A_\nu) \tag{2}$$

leads to the same equations of motion, and find $\mathcal{L} - \mathcal{L}_1$. Why is it that $\mathcal{L}$ and $\mathcal{L}_1$ give rise to the same field equations? (*1.5 points*)

(c) By Legendre transformation, compute the Hamiltonian density $\mathcal{H}$ associated with (2) and express it as function of the fields A_μ and their canonically conjugate momenta. (*1 point*)

(d) For a general Lagrangian $\mathcal{L}$, derive the conserved current of Noether's theorem from the condition of the quasi-invariance of $\mathcal{L}$ under a continuous global symmetry. Next, verify the quasi-invariance of the *specific* Lagrangian (2) under infinitesimal time translations and compute the associated conserved charge. Show that the result coincides with what you got in (c). (*2.5 points*)

(e) Determine the propagator in momentum space of the free field theory defined by (2). (*1.5 points*)

(f) Given the expansion of the vector field in a plane wave basis, we can do the same for the corresponding quantum field operator, namely,

$$\hat{A}_\mu(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left(\hat{a}_A(\vec{k})\epsilon^A_\mu(\vec{k}) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \hat{a}^\dagger_B(\vec{k})\epsilon^{*B}_\mu(\vec{k}) e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \right), \tag{3}$$

where $\omega_k^2 = \vec{k}^2$ and for each momentum mode $\vec{k}$ we have introduced four polarization four-vectors ϵ^A_μ, labelled by $A = 1, ..., 4$, and satisfying the orthogonality relation $\epsilon^{*A}_\mu(\vec{k})\epsilon^{\mu B}(\vec{k}) = \eta^{AB}$, where η is the "Minkowski metric" with respect to the index labels A, B, that is, $\eta^{AB} = \mathrm{diag}(1, -1, -1, -1)$. Compute the quantum Hamiltonian $\hat{H}$ by substituting (3) into the expression for the Hamiltonian $\mathcal{H}$ obtained earlier. Bring the result into the form

$$\hat{H}(x) = \int d^3k \, \eta^{AB} \left(f_1(\vec{k})\hat{a}^\dagger_A(\vec{k})\hat{a}_B(\vec{k}) + f_2(\vec{k})\hat{a}_A(\vec{k})\hat{a}^\dagger_B(\vec{k}) \right), \tag{4}$$

and determine the functions $f_i(\vec{k})$. *(3.5 points)*

Problem 2 - Green's functions (5 points)

Consider the Green's functions

$$D_\alpha = \int \frac{dE}{2\pi} \frac{e^{-iEt}}{E^2 - \omega^2 - i\epsilon E}, \quad D_\rho = \int \frac{dE}{2\pi} \frac{e^{-iEt}}{E^2 - \omega^2 + i\epsilon E}, \quad \epsilon > 0, \tag{5}$$

and evaluate them using the method of residues. What is the associated differential operator? What boundary conditions of the solutions of the differential equation are associated with the two different Green's functions?

Problem 3 - Scattering of scalar particles (4 points)

Compute the amputated amplitudes of all connected diagrams for four-point functions with fixed external four-momenta k_i, $i = 1, ..., 4$ at tree level in the theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}[(\partial\psi)^2 - \mu^2\psi^2)] - \frac{\lambda_3}{3!}\psi^3 - \frac{\lambda_4}{4!}\psi^4, \tag{6}$$

for a real scalar field $\psi(x)$. It is not required that you derive the Feynman rules from an explicit expansion of the path integral.