EXAM BLOCK 2 GEOPHYSICAL FLUID DYNAMICS

17 March 2010, 9.00 - 11.00 hours

Two problems (all items have equal weight)

Remark 1: answers may be written in English or Dutch.

Remark 2: in all questions you may use $g = 10$ ms^{-2}, $a = 6400$ km and $\Omega = 7.3\times10^{-5}$ s^{-1}.

Problem 1

A fluid system is governed by the following equations:

$$\frac{\partial u}{\partial t} - \beta_0\, y\, v = g'\frac{\partial a}{\partial x},$$
$$\frac{\partial v}{\partial t} + \beta_0\, y\, u = g'\frac{\partial a}{\partial y},$$
$$-\frac{\partial a}{\partial t} + H\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = 0.$$

a. Name the set of these equations.
Also, describe the meaning of the symbols β_0, g', a and H.

b. When focusing on flow that evolves on timescales of many days, the system above can be reduced to a single equation for variable v, i.e.,

$$\frac{\partial}{\partial t}\left(\nabla^2 v - \frac{\beta_0^2\, y^2}{g'H}\, v\right) + \beta_0 \frac{\partial v}{\partial x} = 0,$$

which admits wave-like solutions of the form

$$v = \Re\left\{V(y)\, e^{i(kx-\omega t)}\right\}.$$

Derive the equation for $V(y)$.

c. The equation for $V(y)$ found in item b has a solution

$$V = V_0\, y\, \exp\left(-\frac{1}{2}\,\mu^2\, y^2\right), \qquad \mu^2 = \frac{\beta_0}{\sqrt{g'H}},$$

with V_0 a constant and the corresponding dispersion relation

$$\omega = \frac{-\beta_0\, k}{k^2 + 3\mu^2}.$$

What kind of waves are described by this solution?
Also, give a physical interpretation of parameter μ.

d. Compute the range of wavenumbers k for which the phase velocity and group velocity of these waves are of opposite sign.

For problem 2: P.T.O.

Problem 2

Consider flow governed by

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0\,,$$

$$q = \nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0\, y\,.$$

a. What principle is expressed by this equation?

b. Give a physical interpretation of the three terms that constitute variable q.

c. Show that the equation above admits a solution that represents a steady, zonal flow with an arbitrary vertical structure.

d. Assume a steady zonal flow in the atmosphere, which is governed by the equation given above, and which has a constant vertical shear $\alpha = 10^{-3}\ \mathrm{s}^{-1}$. Compute the magnitude and direction of the thermal wind between vertical levels $z = 1$ km and $z = 5$ km.

e. Compute the density field that corresponds to the flow considered in item d.

END

GFD 2009 Equation sheet

Continuity and momentum equations: molecular viscous fluid

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w) = 0$$

$$\begin{aligned}
\rho\left(\frac{du}{dt} + f_\star\, w - f\, v\right) &= -\frac{\partial p}{\partial x} & &+ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
\rho\left(\frac{dv}{dt} + f\, u\right) &= -\frac{\partial p}{\partial y} & &+ \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
\rho\left(\frac{dw}{dt} - f_\star\, u\right) &= -\frac{\partial p}{\partial z} & -\rho g &+ \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}$$

Energy budget for adiabatic flow of fixed composition

$$\rho\, C_v \frac{dT}{dt} - \frac{T}{\rho}\left(\frac{\partial p}{\partial T}\right)_\rho \frac{d\rho}{dt} = 0$$

Relative circulation and relative vorticity

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{u})\cdot \mathbf{n}\, dS$$

where S is the surface enclosed by contour C.

Shallow water equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f\, v &= -g\frac{\partial \eta}{\partial x}\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f\, u &= -g\frac{\partial \eta}{\partial y}\\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0
\end{aligned}$$

Ekman pump (Northern Hemisphere)

$$\overline{w} = \frac{d}{2}\overline{\zeta}, \qquad d = \left(\frac{2\nu_E}{f}\right)^{1/2} \qquad \text{and} \qquad w_{\mathrm{Ek}} = \frac{1}{\rho_0 f}\left[\frac{\partial \tau^y}{\partial x} - \frac{\partial \tau^x}{\partial y}\right]$$

Equation sheet 2

- Shallow water equations for 2 layer model (linear, $f = f_0 + \beta_0 y$):

$$\frac{\partial u_1}{\partial t} - f v_1 = -g \frac{\partial \eta}{\partial x} \qquad \frac{\partial v_1}{\partial t} + f u_1 = -g \frac{\partial \eta}{\partial y}$$

$$\frac{\partial u_2}{\partial t} - f v_2 = -g \frac{\partial \eta}{\partial x} - g' \frac{\partial a}{\partial x} \qquad \frac{\partial v_2}{\partial t} + f u_2 = -g \frac{\partial \eta}{\partial y} - g' \frac{\partial a}{\partial y}$$

$$\frac{\partial}{\partial t}(\eta - a) + \frac{\partial}{\partial x}(H_1 u_1) + \frac{\partial}{\partial y}(H_1 v_1) = 0$$

$$\frac{\partial a}{\partial t} + \frac{\partial}{\partial x}\left[(H_2 - b) u_2\right] + \frac{\partial}{\partial y}\left[(H_2 - b) v_2\right] = 0$$

Characteristic depth: $\bar{h} = \dfrac{H_1 H_2}{H_1 + H_2}$

- Generalised equation for barotropic planetary waves / topographic waves:

$$\frac{\partial \eta}{\partial t} - R^2 \frac{\partial}{\partial t} \nabla^2 \eta - \beta_0 R^2 \frac{\partial \eta}{\partial x} + \frac{\alpha_0 g}{f_0} \frac{\partial \eta}{\partial x} = 0$$

- Complex variable: $\phi \equiv |\phi| e^{i\theta} \equiv \phi_r + i\phi_i$ where $|\phi|^2 = \phi_r^2 + \phi_i^2$, $\tan\theta = \dfrac{\phi_i}{\phi_r}$

- QG Theory for continuously stratified fluid:

$$\frac{\partial w}{\partial z} = \frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p' + \frac{1}{\rho_0 f_0} J(p', \nabla^2 p') + \beta_0 \frac{\partial p'}{\partial x}\right]$$

$$\frac{\partial \rho'}{\partial t} + \frac{1}{\rho_0 f_0} J(p', \rho') - \frac{\rho_0 N^2}{g} w = 0$$

$$N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$$

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \qquad q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y$$

- 2 layer QG model:

$$q_1 = \nabla^2\psi_1 + \frac{1}{2R^2}(\psi_2 - \psi_1) + \beta_0 y \qquad R = \frac{\sqrt{g'H}}{2 f_0}$$

$$q_2 = \nabla^2\psi_2 - \frac{1}{2R^2}(\psi_2 - \psi_1) + \beta_0 y \qquad N^2 = \frac{2g'}{H}$$

$$w = \frac{2 f_0}{N^2 H}\left[\frac{\partial}{\partial t}(\psi_2 - \psi_1) + J(\psi_1, \psi_2)\right]$$

$$a = \frac{f_0}{g}(\psi_2 - \psi_1)$$

$$\rho' = \frac{1}{g}\frac{(\psi_2 - \psi_1)}{H}$$

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